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Vacuum Polarization Induced by a Uniformly Accelerated Charge

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We consider a point charge fixed in the Rindler coordinates which describe a uniformly accelerated frame. We determine an integral expression of the induced charge density due to the vacuum polarization at the first order in the fine structure constant. In the case where the acceleration is weak, we give explicitly the induced electrostatic potential.

1. INTRODUCTION

There has been much interest for a long time in the study of classical and quantum problems in a uniformly accelerated frame. For example, quantum field theory in such a frame yields the Unruh effect. The present paper is concerned with the vacuum polarization due to a charge fixed in a uniformly accelerated frame. So far as we know, the induced vector potential has not been determined in this case.

When the pair creation is neglected, the induced current $\langle j^{\mu} \rangle$ resulting from the vacuum polarization in an external current j^{μ} was determined by Serber (1935) at the first order in the fine structure constant α, making use of the Fourier transform. Schwinger (1949) gave an equivalent integral expression with the aid of the half-sum of advanced and retarded Green functions $\overline{\Delta}(x, x')$. However, the direct application of these formulas to the case of the current of a uniformly accelerated charge seems too difficult.

It is of course natural to analyze this problem in a uniformly accelerated frame described by the Rindler coordinates in which the charge appears as fixed. In consequence, one should infer that there exists only an induced charge density in this frame resulting from the vacuum polarization. Unfortu-

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nately, the above formulas cannot be covariantly written down. However, the Schwinger formula giving the induced current can be developed in power series in $1/m^2$, m being the mass of the electron ($\hbar = c = 1$), and this series may be rewritten in a manifestly covariant manner. As a consequence of this, we will derive an integral expression for the induced charge density with the aid of the Green function for a certain operator expressed in the Rindler coordinates.

The plan of the work is as follows. In Section 2, we recall the basic results of the vacuum polarization in the first order in α. Then we obtain in Section 3 the covariant expression of the induced current in the form of a power series in $1/m^2$. We specialize this result for the case of the Rindler coordinates in Section 4. In Section 5, from this we deduce an integral expression of the induced electrostatic potential for a fixed point charge. In Section 6 we add some concluding remarks.

2. SCHWINGER'S FORMULA

In inertial coordinates (x^0, x^i) the Minkowskian metric has the expression

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \tag{1}$$

The Maxwell equations for the vector potential A_μ are

$$\Box A_\mu = j_\mu \qquad \text{with} \quad \partial_\mu A^\mu = 0 \tag{2}$$

where j^μ is the current which is conserved. We now introduce the half-sum of advanced and retarded Green functions $\overline{\Delta}(x, x')$ for the equation

$$(\Box_x - m^2)\overline{\Delta} = -\delta^{(4)}(x, x') \tag{3}$$

It has the explicit expression

$$\overline{\Delta}(x, x') = \frac{\delta(\lambda)}{4\pi} - \frac{m^2}{8\pi}\begin{cases} J_1(m\lambda^{1/2})/m\lambda^{1/2} & \text{for} \quad \lambda > 0 \\ 0 & \text{for} \quad \lambda < 0 \end{cases} \tag{4}$$

where $\lambda = (x^0 - x'^0)^2 - (x^1 - x'^1)^2 - (x^2 - x'^2)^2 - (x^3 - x'^3)^2$, J_n being the Bessel functions.

Schwinger (1949) showed that the induced current $\langle j^\mu \rangle$ due to the vacuum polarization in an external current j^μ can be calculated by an integral containing $\overline{\Delta}(x, x')$; he found

$$\langle j_\mu(x) \rangle = -\frac{4\alpha}{\pi} \int dx'^0 dx'^1 dx'^2 dx'^3 \int_0^1 dv \, \overline{\Delta}\left[\frac{2}{(1 - v^2)^{1/2}}(x - x')\right] \times \frac{1 - v^2/3}{(1 - v^2)^2} v^2 \Box_{x'} j_\mu(x') \tag{5}$$

The induced vector potential $\langle A_\mu \rangle$ is then determined from Maxwell's equations (2) with current (5); we have immediately

$$\langle A_\mu(x) \rangle = -\frac{4\alpha}{\pi} \int dx'^0 dx'^1 dx'^2 dx'^3 \int_0^1 dv\, \bar{\Delta}\left[\frac{2}{(1 - v^2)^{1/2}}(x - x')\right] \times \frac{1 - v^2/3}{(1 - v^2)^2} v^2 j_\mu(x') \tag{6}$$

In the case of a point charge at rest in inertial coordinates, the nonvanishing component of the external current j^μ is the charge density j^0,

$$j^0(x^i) = e\delta(x^1)\delta(x^2)\delta(x^3) \tag{7}$$

where e is the charge. The corresponding electrostatic potential A_0 is

$$A_0(x^i) = \frac{e}{4\pi r} \tag{8}$$

where $r = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2}$. With a static external source, one can perform the integration with respect to the variable x'^0 in Schwinger's formula (5). So, the useful quantity is now the Green function for the operator $\Delta - m^2$. Finally formula (6), giving the induced electrostatic potential $\langle A_0 \rangle$ for current (7), reduces to

$$\langle A_0(x^i) \rangle = \frac{e}{4\pi r}\frac{\alpha}{\pi} \int_0^1 dv \exp\left[-\frac{2mr}{(1 - v^2)^{1/2}}\right] \frac{1 - v^2/3}{1 - v^2} v^2 \tag{9}$$

This modification of the Coulomb law has a range $1/2m$. The determination of $\langle A_0 \rangle$ has been done by Uehling (1935), but we give the expression in closed form found by Pauli and Rose (1936). We set

$$\langle A_0(r) \rangle = \frac{e}{4\pi r} U(mr) \tag{10}$$

where the function U can be expressed in terms of elementary functions

$$U(z) = \frac{\alpha}{3\pi}\left[2\left(\frac{z^2}{3} + 1\right)K_0(2z) - \frac{2z}{3}(2z^2 + 5)K_1(2z) + z\left(\frac{4z^2}{3} + 3\right)Ki_1(2z)\right] \tag{11}$$

K_n are the modified Bessel functions of the second kind and Ki_n the repeated integrals of K_0. From expression (11) we obtain easily the asymptotic form of U for small values of z

$$U(z) \sim -\frac{2\alpha}{3\pi}\left(\gamma + \frac{5}{6} + \ln z\right) \tag{12}$$

where γ is Euler's constant.

In the case of a point charge which is uniformly accelerated with an acceleration g, the external current j^{μ} has the components

$$\begin{aligned} j^0(x) &= e\delta\{x^1 - [1/g^2 + (x^0)^2]^{1/2}\}\delta(x^2)\delta(x^3) \\ j^1(x) &= e\,\frac{x^0}{[1/g^2 + (x^0)^2]^{1/2}}\,\delta\{x^1 - [1/g^2 + (x^0)^2]^{1/2}\}\delta(x^2)\delta(x^3) \\ j^2(x) &= j^3(x) = 0 \end{aligned} \tag{13}$$

The application of Schwinger's formula (5) for the current (13) is possible in principle, but the actual calculations are too complicated.

3. COVARIANT FORMULA IN POWER SERIES IN $1/m^2$

Schwinger's formula (5) can be developed in power series in $1/m^2$. According to equation (3), we have the relation

$$\begin{aligned} \overline{\Delta}\left[\frac{2}{(1 - v^2)^{1/2}}\,x\right] &= \frac{(1 - v^2)^2}{16m^2}\,\delta^{(4)}(x) \\ &\quad + \frac{1 - v^2}{4m^2}\,\Box\overline{\Delta}\left[\frac{2}{(1 - v^2)^{1/2}}\,x\right] \end{aligned} \tag{14}$$

By inserting (14) into (5) and by performing an integration by parts, we obtain

$$\begin{aligned} \langle j_{\mu}(x)\rangle &= -\frac{\alpha}{4\pi m^2}\int_0^1 dv\left(1 - \frac{v^2}{3}\right)v^2\,\Box j_{\mu}(x) \\ &\quad - \frac{\alpha}{\pi m^2}\int dx'^0dx'^1dx'^2dx'^3\int_0^1 dv\,\overline{\Delta}\left[\frac{2}{(1 - v^2)^{1/2}}\,(x - x')\right] \\ &\quad \times \frac{1 - v^2/3}{1 - v^2}\,v^2\Box^2_{x'}j_{\mu}(x') \end{aligned} \tag{15}$$

The term proportional to $\Box j_{\mu}$ in (15) is the first term of the power series in $1/m^2$. By inserting again relation (14) into the second term in (15), we will obtain the second term of the power series in $1/m^2$, and so on. Hence the current of polarization has the expression

$$\langle j_\mu(x)\rangle = \sum_{n=1}^{\infty} a_n \frac{1}{m^{2n}} \Box^n j_\mu(x) \tag{16}$$

where all the coefficients a_n can be calculated. In particular we have

$$a_1 = -\frac{\alpha}{15\pi m^2} \tag{17}$$

Each term of the series (16) is a 1-form that we may covariantly write down. We now consider an arbitrary coordinate system $(x^{\mu'})$ of the Minkowski spacetime. The components of the Minkowskian metric are denoted $g_{\rho'\sigma'}$ and the covariant derivative $\nabla_{\rho'}$. According to (16), the induced current $\langle j_{\mu'}\rangle$ can be expressed as a function of the external current $j_{\mu'}$ by the following power series in $1/m^2$:

$$\langle j_{\mu'}(x')\rangle = \sum_{n=1}^{\infty} a_n \frac{1}{m^{2n}} (g^{\rho'\sigma'}\nabla_{\rho'}\nabla_{\sigma'} j_{\mu'})^n \tag{18}$$

where the operators in (18) are defined by the recurrence law

$$(g^{\rho'\sigma'}\nabla_{\rho'}\nabla_{\sigma'} j_{\mu'})^n = g^{\rho'\sigma'}\nabla_{\rho'}\nabla_{\sigma'}[(g^{\rho'\sigma'}\nabla_{\rho'}\nabla_{\sigma'} j_{\mu'})^{n-1}] \qquad (n \geq 1) \tag{19}$$

To establish this result we have taken into account that the Ricci tensor vanishes.

The induced vector potential $\langle A_{\mu'}\rangle$ satisfies the covariant Maxwell equations

$$g^{\rho'\sigma'}\nabla_{\rho'}\nabla_{\sigma'} A_{\mu'} = j_{\mu'} \qquad \text{with} \quad \nabla_{\mu'} A^{\mu'} = 0 \tag{20}$$

4. CASE OF THE RINDLER COORDINATES

The application of Schwinger's formula (5) to the case of a uniformly accelerated charge, described by current (13), seems very difficult because the problem is time-dependent. But we know that the Rindler coordinates $(\xi^0, \xi^1, \xi^2, \xi^3)$ with $\xi^1 > 0$ describe a uniformly accelerated frame in the Minkowski spacetime. For an acceleration g, the coordinate transform from inertial coordinates is

$$\begin{aligned} x^0 &= \xi^1 \sinh(g\xi^0) \\ x^1 &= \xi^1 \cosh(g\xi^0) \\ x^2 &= \xi^2 \\ x^3 &= \xi^3 \end{aligned} \tag{21}$$

In this coordinate system, Minkowskian metric (1) takes the form

$$ds^2 = -g^2(\xi^1)^2(d\xi^0)^2 + (d\xi^1)^2 + (d\xi^2)^2 + (d\xi^3)^2 \tag{22}$$

The charge having an acceleration g now will be located at the point

$$\xi^1 = \frac{1}{g} \qquad \xi^2 = \xi^3 = 0 \tag{23}$$

and its current (13) has the following components:

$$\begin{aligned} j^{\xi^0} &= e\delta\left(\xi^1 - \frac{1}{g}\right)\delta(\xi^2)\delta(\xi^3) \\ j^{\xi^1} &= 0 \\ j^{\xi^2} &= j^{\xi^3} = 0 \end{aligned} \tag{24}$$

in the Rindler coordinates. Consequently, the uniformly accelerated charge is described by a point charge at rest.

Maxwell's equations (20) written in the Rindler coordinates for a static charge density j^{ξ^0} reduce to an equation for the electrostatic potential A_{ξ^0},

$$\left(\Delta_\xi - \frac{1}{\xi^1}\frac{\partial}{\partial \xi^1}\right)A_{\xi^0} = j_{\xi^0} \tag{25}$$

where $j_{\xi^0} = -(g\xi^1)^2 j^{\xi^0}$. For a point charge at rest, j^{ξ^0} being given by (24), the electrostatic potential V_W was found by Whittaker (1927), in a slightly different coordinate system, which corresponds to the retarded solution to the Maxwell equations with current (13).

In Rindler coordinates, we remark that the operator $g^{\rho\sigma}\nabla_\rho\nabla_\sigma j_\mu$ applied to a static charge density j^{ξ^0} is simple since we have

$$\begin{aligned} (g^{\rho\sigma}\nabla_\rho\nabla_\sigma j_{\xi^0}) &= \left(\nabla_\xi - \frac{1}{\xi^1}\frac{\partial}{\partial \xi^1}\right)j_{\xi^0} \\ (g^{\rho\sigma}\nabla_\rho\nabla_\sigma j_{\xi^1}) &= (g^{\rho\sigma}\nabla_\rho\nabla_\sigma j_{\xi^2}) = (g^{\rho\sigma}\nabla_\rho\nabla_\sigma j_{\xi^3}) = 0 \end{aligned} \tag{26}$$

As a consequence of properties (26), expression (18) of the induced current yields only a charge density $\langle j^{\xi^0}\rangle$. This fact is natural since now we have a problem which does not depend on the time. By defining the operator

$$\mathcal{D}_\xi = \Delta_\xi - \frac{1}{\xi^1}\frac{\partial}{\partial \xi^1} \tag{27}$$

we can rewrite the induced charge density as a power series in $1/m^2$,

$$\langle j_{\xi^0} \rangle = \sum_{n=1}^{\infty} a_n \frac{1}{m^{2n}} \mathcal{D}_\xi^n j_{\xi^0} \tag{28}$$

where j^{ξ^0} is the external charge density. Maxwell's equations (25) are also rewritten in the form

$$\mathcal{D}_\xi \langle A_{\xi^0} \rangle = \langle j_{\xi^0} \rangle \tag{29}$$

In the Rindler coordinates, the first correction to the Whittaker potential which is necessary to evaluate the Lamb shift is

$$\langle A_{\xi^0}(\xi^i) \rangle = e \frac{\alpha}{15\pi m^2} \delta\left(\xi^1 - \frac{1}{g}\right) \delta(\xi^2) \delta(\xi^3) \tag{30}$$

since a_1 has value (17).

5. VACUUM POLARIZATION FOR A CHARGE FIXED IN THE RINDLER COORDINATES

We now define the Green function $\mathcal{G}(\xi^i, \xi'^i)$ for the equation

$$(\mathcal{D}_\xi - m^2)\mathcal{G} = -g\xi^1 \delta^{(3)}(\xi^i - \xi'^i) \tag{31}$$

assuming that $\mathcal{G}(\xi^i, \xi'^i)$ vanishes when the points ξ^i and ξ'^i are infinitely separated. Now the operator $1/\xi \mathcal{D}_\xi$ is self-adjoint, and therefore the Green function is symmetric and satisfies the identities

$$\mathcal{D}_\xi \mathcal{G}(\xi^i, \xi'^i) = \mathcal{D}_{\xi'} \mathcal{G}(\xi^i, \xi'^i)$$

$$\int d\xi^1 d\xi^2 d\xi^3 f(\xi^i) \frac{1}{\xi^1} \mathcal{D}_\xi g(\xi^i) = \int d\xi^1 d\xi^2 d\xi^3 g(\xi^i) \frac{1}{\xi^1} \mathcal{D}_\xi f(\xi^i) \tag{32}$$

where f and g are two arbitrary functions.

For a static charge density j^{ξ^0} we are now in a position to give the formula giving the induced charge density $\langle j^{\xi^0} \rangle$ under an integral form:

$$\begin{aligned} \langle j_{\xi^0}(\xi^i) \rangle = -\frac{\alpha}{\pi} \int d\xi'^1 d\xi'^2 d\xi'^3 \\ \times \int_0^1 dv\, \mathcal{G}\left[\frac{2}{(1-v^2)^{1/2}} \xi^i, \frac{2}{(1-v^2)^{1/2}} \xi'^i\right] \\ \times \frac{1 - v^2/3}{1 - v^2} v^2 \frac{1}{g\xi'^1} \mathcal{D}_{\xi'} j_{\xi^0}(\xi') \end{aligned} \tag{33}$$

In order to prove this, we develop formula (33) in a power series in $1/m^2$. We proceed as in Section 3. From (31) we have the relation

$$\mathcal{G}\left[\frac{2}{(1-v^2)^{1/2}}\xi^i, \frac{2}{(1-v^2)^{1/2}}\xi'^i\right] = \frac{1-v^2}{4m^2} g\xi^1\delta^{(3)}(\xi^i - \xi'^i) + \frac{1-v^2}{4m^2}\mathcal{D}_\xi\mathcal{G}\left[\frac{2}{(1-v^2)^{1/2}}\xi^i, \frac{2}{(1-v^2)^{1/2}}\xi'^i\right] \tag{34}$$

taking into account the specific property of the operator $\mathcal{D}_\xi$. By inserting (34) into (33) and making use of the identities (32), we find

$$\langle j_{\xi^0}(\xi^i)\rangle = -\frac{\alpha}{4\pi m^2}\int_0^1 dv\left(1-\frac{v^2}{3}\right)v^2\,\mathcal{D}_\xi j_{\xi^0}(\xi^i) - \frac{\alpha}{4\pi m^2}\int d\xi'^1 d\xi'^2 d\xi'^3 \times \int_0^1 dv\,\mathcal{G}\left[\frac{2}{(1-v^2)^{1/2}}\xi^i, \frac{2}{(1-v^2)^{1/2}}\xi'^i\right] \times \left(1-\frac{v^2}{3}\right)v^2\frac{1}{g\xi'^1}\mathcal{D}^2_{\xi'} j_{\xi^0}(\xi'^i) \tag{35}$$

We recognize that the coefficient in front of $\mathcal{D}_\xi j_{\xi^0}$ is a_1. The repeated applications of relation (34) yield the power series (28). Therefore, we conclude that formula (33) gives the induced charge density due to the vacuum polarization. However, we are not worried about the boundary conditions of the Green function $\mathcal{G}(\xi^i, \xi'^i)$ at the hypersurface $\xi^1 = 0$ in the metric (22).

We obtain the induced electrostatic potential from Maxwell's equations (29) by using again (32),

$$\langle A_{\xi^0}(\xi^i)\rangle = -\frac{\alpha}{\pi}\int d\xi'^1 d\xi'^2 d\xi'^3 \times \int_0^1 dv\,\mathcal{G}\left[\frac{2}{(1-v^2)^{1/2}}\xi^i, \frac{2}{(1-v^2)^{1/2}}\xi'^i\right] \times \frac{1-v^2/3}{1-v^2}v^2\frac{1}{g\xi'^1} j_{\xi^0}(\xi'^i) \tag{36}$$

In the case of charge density (24), the integral (36) reduces to

$$\langle A_{\xi^0}(\xi^i)\rangle = \frac{e\alpha}{\pi}\int_0^1 dv\, \mathcal{G}\left[\frac{2}{(1-v^2)^{1/2}}\,\xi^i, \frac{2}{(1-v^2)^{1/2}}\,\xi_g^i\right]\frac{1-v^2/3}{1-v^2}\,v^2 \tag{37}$$

where $\xi_g^i = (1/g, 0, 0)$. By the inverse transform of coordinates (21) we can express the induced vector potential $\langle A_\mu\rangle$ in inertial coordinates.

The problem is now to determine the Green function $\mathcal{G}(\xi^i, \xi'^i)$. We restrict ourselves to finding an expression at the first order in g. To do this, we introduce the new coordinates

$$y^1 = \xi^1 - \frac{1}{g}, \qquad y^2 = \xi^2, \qquad y^3 = \xi^3 \tag{38}$$

With variables (38), equation (31) takes the form

$$\left(\Delta_y - \frac{g}{1+gy^1}\frac{\partial}{\partial y^1} - m^2\right)\mathcal{G} = -(1+gy^1)\delta^{(3)}(y^i - y'^i) \tag{39}$$

and, by keeping the terms linear in g, this becomes

$$\left(\Delta_y - g\frac{\partial}{\partial y^1} - m^2\right)\mathcal{G} = -(1+gy^1)\delta^{(3)}(y^i - y'^i) \tag{40}$$

The domain of validity of equation (40) is restricted to $gy^1 << 1$. We choose the solution of this equation which reduces to the Green function for the operator $\Delta - m^2$ in the limit where g tends to 0. We do not touch upon the problem of the global definition of the Green function $\mathcal{G}(\xi^i, \xi'^i)$. We find

$$\mathcal{G}(y^i, y'^i) = \frac{\exp - m|y^i - y'^i|}{4\pi|y^i - y'^i|}\left(1 + \frac{1}{2}gy^1 + \frac{1}{2}gy'^1\right) + O(g^2) \tag{41}$$

We are now in a position to calculate formula (37) at the first order in g. We have to perform $\xi \rightsquigarrow 2/(1-v^2)^{1/2}\xi$, which we write in the coordinates y^i,

$$1 + \frac{1}{2}gy^1 + \frac{1}{2}gy'^1 \rightsquigarrow \frac{2 + gy^1 + gy'^1}{(1-v^2)^{1/2}}$$

So, we obtain

$$\langle A_{\xi^0}(\xi^i)\rangle \approx \frac{e}{4\pi\epsilon}\frac{\alpha}{\pi}\int_0^1 dv\,\exp\left[-\frac{2m\epsilon}{(1-v^2)^{1/2}}\right]\frac{1-v^2/3}{1-v^2}\,v^2\left(1+\frac{1}{2}gy^1\right) \tag{42}$$

where $\epsilon = [(y^1)^2 + (y^2)^2 + (y^3)^2]^{1/2}$. According to expression (9) of the Uehling potential $\langle A_0\rangle$, we can write

$$\langle A_{\xi^0}(y^i)\rangle \approx \langle A_0(y^i)\rangle + \frac{1}{2}g\langle A_0(y^i)\rangle y^1 \tag{43}$$

At the first order in g, the metric (22) takes the form

$$ds^2 \approx -(1 + 2gy^1)(dy^0)^2 + (dy^1)^2 + (dy^2)^2 + (dy^3)^2 \tag{44}$$

which is valid for $gy^1 << 1$. The Whittaker potential is then approximated by

$$V_W(y^i) \approx \frac{e}{4\pi\epsilon}\left(1 + \frac{1}{2}gy^1\right) \tag{45}$$

The total electrostatic potential V, the sum of (43) and (45), generated by a point charge located at $y^i = 0$ in the metric (44), taking into account the vacuum polarization at the first order in α, is given by the expression

$$V(y^i) \approx \frac{e}{4\pi\epsilon}\left[1 + U(m\epsilon) + \frac{1}{2}gy^1 + \frac{1}{2}gU(m\epsilon)y^1\right] \tag{46}$$

where U is given by (11).

6. CONCLUSION

We have given an integral expression (37) for the induced electrostatic potential with the aid of the Green function for the operator (31) in the Rindler coordinates. This determination is obtained from the Schwinger formula in inertial coordinates. However, it would be of conceptual interest to derive directly this formula within the framework of quantum electrodynamics in Rindler spacetime in order to discuss the effect of the horizon.

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